Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel Level 3 GCE

Wednesday 13 October 2021 – Afternoon

Time 2 hours

Paper reference **9MA0/02**

Mathematics

Advanced
PAPER 2: Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 15 questions in this question paper. The total mark for this paper is 100.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

Pearson

1. In an arithmetic series

- the first term is 16
- the 21st term is 24

(a) Find the common difference of the series.

(2)

(b) Hence find the sum of the first 500 terms of the series.

(2)

General formula for n^{th} term U_n of arithmetic series:

$$U_n = a + (n-1)d$$

Given, $a = 16$

$U_{21} = 24$

a) using formula: $U_{21} = a + (n-1)d$

$$24 = 16 + (21-1)d$$

$$\Rightarrow 24 = 16 + 20d$$

$$\Rightarrow 20d = 8$$

$$\Rightarrow d = \frac{2}{5}$$

b) sum for n terms $= \frac{n}{2}\left[2a + (n-1)d\right]$

$$S_{500} = \frac{500}{2}\left[2\times 16 + (500-1)d\right]$$

$$= 250 \times \left[32 + 499 \times \frac{2}{5}\right]$$

$$= 250 \times \frac{1158}{5}$$

$$= 57900$$

2. The functions f and g are defined by

$$f(x) = 7 - 2x^2 \qquad x \in \mathbb{R}$$

$$g(x) = \frac{3x}{5x-1} \qquad x \in \mathbb{R} \qquad x \neq \frac{1}{5}$$

(a) State the range of f

(1)

(b) Find gf(1.8)

(2)

(c) Find $g^{-1}(x)$

(2)

a) $f(x) = 7 - 2x^2$

max. value of $-2x^2 = 0$.
since $x^2 \geqslant 0$

$\therefore$ max. value of $7 - 2x^2$ is $7 - 0$
$= 7$

$\therefore$ the range of f is $f(x) \leq 7$, $f(x) \in R$

b) $gf(x) = \dfrac{3 \times (7 - 2x^2)}{5 \times (7 - 2x^2) - 1}$

$= \dfrac{21 - 6x^2}{35 - 10x^2 - 1}$

$= \dfrac{21 - 6x^2}{34 - 10x^2}$

$\therefore gf(1.8) = \dfrac{21 - 6(1.8)^2}{34 - 10(1.8)^2}$

Question 2 continued

$$= \frac{21 - 6 \times 3.24}{34 - 10 \times 3.24}$$

$$= \frac{1.56}{1.6}$$

$$gf(1.8) = \frac{39}{40}$$

c) To find $g^{-1}(x)$, we must equate $g(x)$ to y :

$$y = \frac{3x}{5x-1}$$

$$\Rightarrow (5x-1)y = 3x$$

$$\Rightarrow 5xy - y = 3x$$

$$\Rightarrow 5xy - 3x = y$$

$$\Rightarrow x(5y-3) = y$$

$$\Rightarrow x = \frac{y}{5y-3}$$

replacing all x's with y's and vice versa :

$$y = \frac{x}{5x-3}$$

$$\therefore g^{-1}(x) = \frac{x}{5x-3}, \quad x \in \mathbb{R}, x \neq \frac{3}{5}$$

(Remember to clearly state the domain)

3. Using the laws of logarithms, solve the equation

$$\log_3 (12y + 5) - \log_3 (1 - 3y) = 2$$

(3)

using $\log a - \log b = \log\left(\frac{a}{b}\right)$,

$\log_3 (12y+5) - \log_3 (1-3y) = \log_3\left(\frac{12y+5}{1-3y}\right)$

$\Rightarrow \quad \log_3\left(\frac{12y+5}{1-3y}\right) = 2$

since $3^2 = 9$, $\log_3 9 = 2$

$\therefore \quad \log_3\left(\frac{12y+5}{1-3y}\right) = \log_3 9$

cancelling logs:

$\frac{12y+5}{1-3y} = 9$

$\Rightarrow \quad 12y + 5 = 9(1-3y)$

$\Rightarrow \quad 12y + 5 = 9 - 27y$

$\Rightarrow \quad 39y = 4$

$\Rightarrow \quad y = \frac{4}{39}$

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4. Given that θ is small and measured in radians, use the small angle approximations to show that

$$4\sin\frac{\theta}{2} + 3\cos^2\theta \approx a + b\theta + c\theta^2$$

where a, b and c are integers to be found.

(3)

$4\sin\frac{\theta}{2} + 3\cos^2\theta$

$= 4\sin\frac{\theta}{2} + 3(1-\sin^2\theta)$ $[\because \sin^2\theta + \cos^2\theta \equiv 1]$

small angle approximation for $\sin\theta$:

$\sin\theta \approx \theta$

$\therefore \sin\frac{\theta}{2} \approx \frac{\theta}{2}$.

$\sin^2\theta \approx \theta^2$

$\Rightarrow 4\sin\frac{\theta}{2} + 3(1-\sin^2\theta) \approx 4\times\frac{\theta}{2} + 3(1-\theta^2)$

$\approx 2\theta + 3 - 3\theta^2$

$\approx 3 + 2\theta - 3\theta^2$

$\therefore a = 3$

$b = 2$

$c = -3$

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P 6 8 7 3 2 A 0 8 4 8

5. The curve C has equation

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11 \qquad x \in \mathbb{R}$$

(a) Find

(i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(b) (i) Verify that C has a stationary point at $x = 1$

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

(4)

a) i. $\dfrac{dy}{dx} = 4 \times 5x^3 - 3 \times 24x^2 + 2 \times 42x - 32$

$= 20x^3 - 72x^2 + 84x - 32$

ii. $\dfrac{d^2y}{dx^2} = 3 \times 20x^2 - 2 \times 72x + 84$

$= 60x^2 - 144x + 84$

b) i. at stationary points, $\dfrac{dy}{dx} = 0$

$20x^3 - 72x^2 + 84x - 32 = 0$

using calculator to find roots :

$x = \dfrac{8}{5}$ or $x = 1$

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Question 5 continued

$\therefore$ at $x = 1$, $\dfrac{dy}{dx} = 0$

$\therefore$ at $x=1$, there is a stationary point

QED

ii. at $x = 1$,

$$\frac{d^2y}{dx^2} = 60 \times 1^2 - 144 \times 1 + 84$$

$$= 144 - 144$$

$$= 0$$

Further inspection is needed.
We need to check the gradient of the curve just before and after $x=1$

$$f'(0.9) = 20 \times (0.9)^3 - 72 \times (0.9)^2 + 84 \times 0.9 - 32$$

$$= -0.14 < 0$$

$$f'(1.1) = 20 \times (1.1)^3 - 72 \times (1.1)^2 + 84 \times 1.1 - 32$$

$$= -0.1 < 0$$

since the gradient remains negative before and after $x=1$, and $\frac{d^2y}{dx^2}$ at $x=1$ is zero,
the stationary point at $x=1$ is a point of inflection.

(Total for Question 5 is 7 marks)

6.

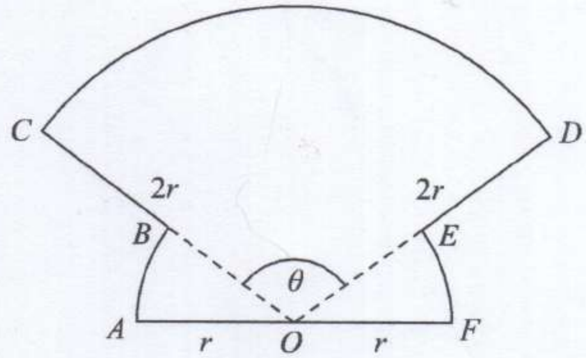


Figure 1

The shape *OABCDEFO* shown in Figure 1 is a design for a logo.

In the design

- *OAB* is a sector of a circle centre *O* and radius r
- sector *OFE* is congruent to sector *OAB*
- *ODC* is a sector of a circle centre *O* and radius $2r$
- *AOF* is a straight line

Given that the size of angle *COD* is θ radians,

(a) write down, in terms of θ, the size of angle *AOB*

(1)

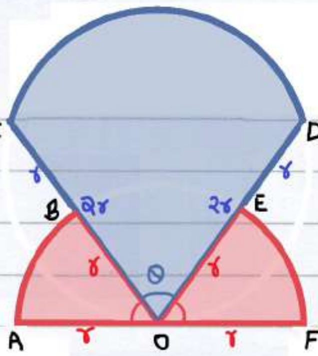
(b) Show that the area of the logo is

$$\frac{1}{2}r^2(3\theta + \pi)$$

(2)

(c) Find the perimeter of the logo, giving your answer in simplest form in terms of r, θ and π.

(2)



∵ it is a straight line,

$\angle AOB + \angle COD + \angle EOF = \pi$

Question 6 continued

but sector OFE is congruent to sector OAB.

$\therefore \angle AOB = \angle EOF$

$\angle AOB + \angle COD + \angle AOB = \pi$

$2\angle AOB + \theta = \pi$

$2\angle AOB = \pi - \theta$

$\angle AOB = \dfrac{\pi - \theta}{2}$

b) Area of a sector $= \dfrac{1}{2} r^2\theta$

For sector OAB & OFE,

area $= \dfrac{1}{2} r^2 \left(\dfrac{\pi-\theta}{2}\right)$

For sector ODC,

area $= \dfrac{1}{2} \times (2r)^2 \times \theta$

$\therefore$ Total area $=$ ODC $+$ OAB $+$ OFE

$= \dfrac{(2r)^2\theta}{2} + \dfrac{r^2}{2}\left(\dfrac{\pi-\theta}{2}\right) + \dfrac{r^2}{2}\left(\dfrac{\pi-\theta}{2}\right)$

$= \dfrac{4r^2\theta}{2} + \dfrac{r^2(\pi-\theta)}{4} + \dfrac{r^2(\pi-\theta)}{4}$

$= 2r^2\theta + \dfrac{2r^2(\pi-\theta)}{4}$

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Question 6 continued

$$= 2r^2\theta + \frac{r^2(\pi-\theta)}{2}$$

$$= \frac{1}{2} r^2 \left(4\theta + \pi - \theta\right)$$

$$= \frac{1}{2} r^2 \left(3\theta + \pi\right)$$

Q.E.D.

(c) Perimeter = OA + AB + BC + CD + DE + EF + FO

length of arc = $r\theta$

$\therefore$ $AB = EF = r\left(\frac{\pi-\theta}{2}\right)$

$CD = 2r\theta$

$CB = ED = 2r - r$
$= r$

$\therefore$ Perimeter $= r + r\left(\frac{\pi-\theta}{2}\right) + r + 2r\theta + r + r\left(\frac{\pi-\theta}{2}\right) + r$

$$= 4r + 2r\theta + \frac{2r(\pi-\theta)}{2}$$

$$= 4r + 2r\theta + r(\pi-\theta)$$

$$= 4r + 2r\theta + r\pi - r\theta$$

$$= 4r + r\pi + r\theta$$

7. **In this question you should show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

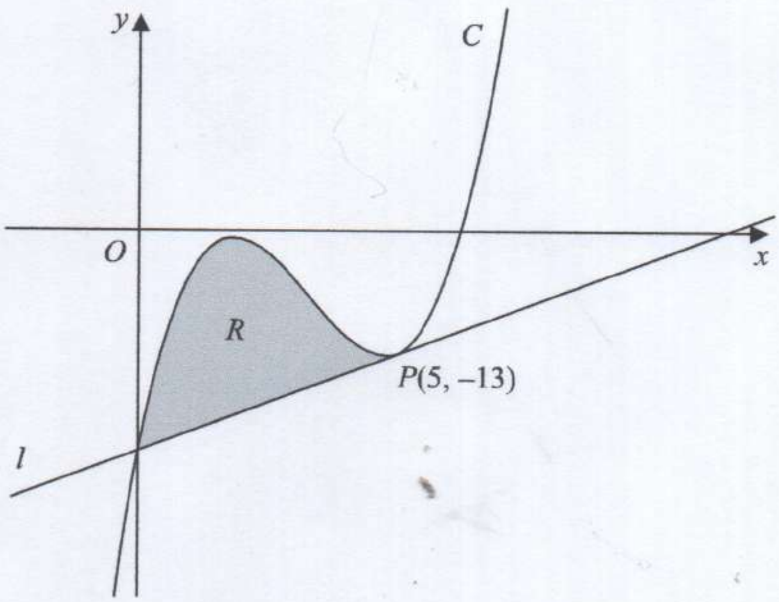


Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = x^3 - 10x^2 + 27x - 23$$

The point $P(5, -13)$ lies on C

The line l is the tangent to C at P

(a) Use differentiation to find the equation of l, giving your answer in the form $y = mx + c$ where m and c are integers to be found.

(4)

(b) Hence verify that l meets C again on the y-axis.

(1)

The finite region R, shown shaded in Figure 2, is bounded by the curve C and the line l.

(c) Use algebraic integration to find the exact area of R.

(4)

(a) To find the gradient at P, we need to differentiate:

$\frac{dy}{dx} = 3x^2 - 20x + 27$

Question 7 continued

when $x = 5$,

$\frac{dy}{dx} = 3 \times 5^2 - 20 \times 5 + 27$

$= 3 \times 25 - 100 + 27$

$= 75 + 27 - 100$

$= 2$

$\therefore$ gradient of tangent $m = 2$ and point $(5, -13)$

using $y - y_1 = m(x - x_1)$:

$y - (-13) = 2(x - 5)$

$y + 13 = 2x - 10$

$y = 2x - 23$

$\therefore m = 2$

$c = -23$

(b) at the y-axis, $x = 0$

substituting $x = 0$ in equation for C :

$y = 0^3 - 10 \times 0 + 27 \times 0 - 23$

$= -23$

substituting $x = 0$ in equation for l:

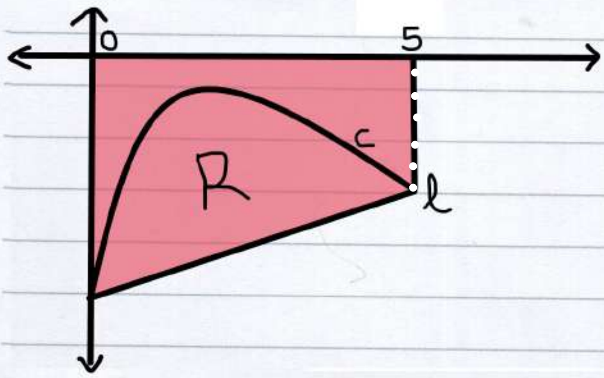
$y = 2 \times 0 - 23$

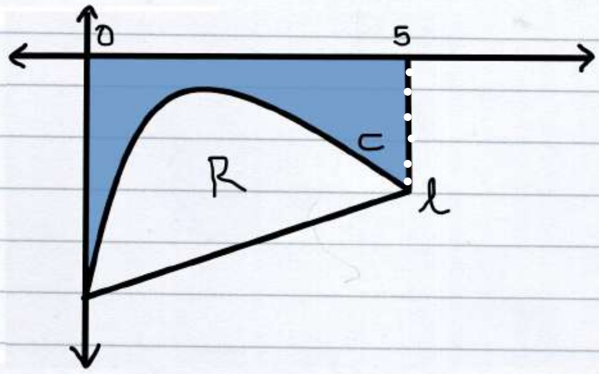
$= -23$

$\therefore$ both l and C pass through the point $(0, -23)$

thus, l meets C on the y-axis

Question 7 continued





(c) Area R = Area bound by l and x-axis – Area bound by C and x-axis

Area bound by l $= \left| \int_0^5 2x - 23 \, .dx \right|$

$= \left| \left[\frac{2x^2}{2} - 23x \right]_0^5 \right|$

$= \left| \left[x^2 - 23x \right]_0^5 \right|$

$= \left| 5^2 - 23 \times 5 - 0 \right|$

$= \left| -90 \right|$

$= 90$

Area bound by C $= \left| \int_0^5 x^3 - 10x^2 + 27x - 23 \, .dx \right|$

$= \left| \left[\frac{x^4}{4} - \frac{10x^3}{3} + \frac{27x^2}{2} - 23x \right]_0^5 \right|$

(Total for Question 7 is 9 marks)

Question 7 continued

$$= \left| \frac{5^4}{4} - \frac{10\times5^3}{3} + \frac{27\times5^2}{2} - 23\times5 - 0 \right|$$

$$= \left| -\frac{455}{12} \right|$$

$$= \frac{455}{12}$$

$\therefore$ Area $R = 90 - \frac{455}{12}$

$= \frac{625}{12}$ sq. units

(Total for Question 7 is 9 marks)

8. The curve C has equation

$$px^3 + qxy + 3y^2 = 26$$

where p and q are constants.

(a) Show that

$$\frac{dy}{dx} = \frac{apx^2 + bqy}{qx + cy}$$

where a, b and c are integers to be found.

(4)

Given that

- the point $P(-1, -4)$ lies on C
- the normal to C at P has equation $19x + 26y + 123 = 0$

(b) find the value of p and the value of q.

(5)

$$px^3 + qxy + 3y^2 = 26$$

a) differentiating with respect to x:

$$3px^2 + \frac{d}{dx}(qxy) + 6y\frac{dy}{dx} = 0$$

using product rule for $\frac{d}{dx}(qxy)$:

$$3px^2 + qx\frac{dy}{dx} + qy + 6y\frac{dy}{dx} = 0$$

$$qx\frac{dy}{dx} + 6y\frac{dy}{dx} = -3px^2 - qy$$

$$\frac{dy}{dx}(qx + 6y) = -3px^2 - qy$$

$$\frac{dy}{dx} = \frac{-3px^2 - qy}{qx + 6y}$$

Question 8 continued

$\therefore \quad a = -3$

$b = -1$

$c = 6$

b) given, at P, when $x = -1$, $y = -4$

substituting x and y in C

$$p(-1)^3 + q(-1)(-4) + 3(-4)^2 = 26$$

$$-p + 4q + 48 = 26$$

$$p - 4q = 22 \qquad -①$$

also given, $19x + 26y + 123 = 0$

$$26y = -19x - 123$$

$$y = \frac{-19}{26}x - \frac{123}{26}$$

$\therefore$ gradient at $P = -1 \div \left(\frac{-19}{26}\right)$

$$= \frac{26}{19}$$

$\therefore \frac{dy}{dx}$ at $P = \frac{26}{19}$

$$\frac{-3p(-1)^2 - q(-4)}{q(-1) + 6(-4)} = \frac{26}{19}$$

Question 8 continued

$$\frac{-3p+4q}{-q-24} = \frac{26}{19}$$

$$19(-3p+4q) = 26(-q-24)$$

$$-57p + 76q = -26q - 624$$

$$57p - 102q = 624 \qquad -② $$

$$① \times 57 : \quad 57p - 4 \times 57 \times q = 22 \times 57$$

$$57p - 228q = 1254 \qquad - ③$$

$$③ - ② : \quad 57p - 228q - (57p - 102q) = 1254 - 624$$

$$57p - 57p - 228q + 102q = 630$$

$$-126q = 630$$

$$q = -5$$

substituting q in ①

$$p - 4(-5) = 22$$

$$p = 22 + 20$$
$$= 42$$

$\therefore$ $q = -5$

$p = 42$

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9. Show that

$$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n \cos(180n)^\circ = \frac{9}{28}$$

(3)

Testing first few terms:

when $n=2$: $\left(\frac{3}{4}\right)^2 \cos(180\times 2)$

$= \left(\frac{3}{4}\right)^2 \cos 360$

$= \left(\frac{3}{4}\right)^2$

$n=3$: $\left(\frac{3}{4}\right)^3 \cos 540$

$= -\left(\frac{3}{4}\right)^3$

$n=4$: $\left(\frac{3}{4}\right)^4 \cos 720$

$= \left(\frac{3}{4}\right)^4$

we can see a pattern emerging, whenever n is odd, $\cos(180n)$ is $=-1$ and whenever n is even, $\cos(180n)=1$

Question 9 continued

$\therefore$ this series can be modelled as a geometric series, where $a = \left(\frac{3}{4}\right)^2$ and $r = -\frac{3}{4}$

$\because \quad |r| < 1$,

we can use S_∞

$$S_\infty = \frac{a}{1-r}$$

$$= \left(\frac{3}{4}\right)^2 \div \left[1 - \left(-\frac{3}{4}\right)\right]$$

$$= \frac{9}{16} \div \left(1 + \frac{3}{4}\right)$$

$$= \frac{9}{16} \div \frac{7}{4}$$

$$= \frac{9}{16} \times \frac{4}{7}$$

$$= \frac{9}{28}$$

Q.E.D.

(Total for Question 9 is 3 marks)

10. The time, T seconds, that a pendulum takes to complete one swing is modelled by the formula

$$T = al^b$$

where l metres is the length of the pendulum and a and b are constants.

(a) Show that this relationship can be written in the form

$$\log_{10} T = b\log_{10} l + \log_{10} a$$

(2)

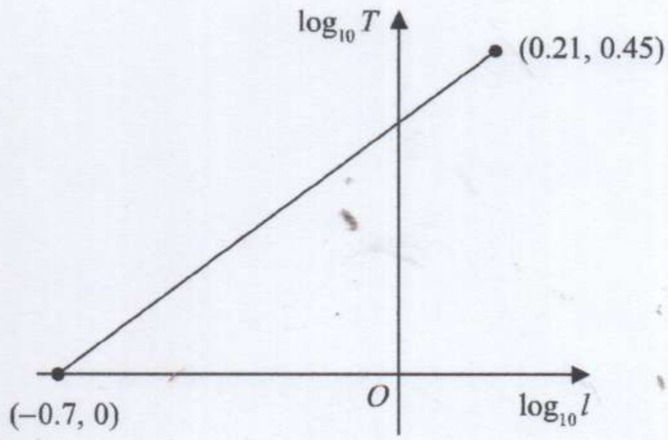


Figure 3

A student carried out an experiment to find the values of the constants a and b.

The student recorded the value of T for different values of l.

Figure 3 shows the linear relationship between $\log_{10} l$ and $\log_{10} T$ for the student's data.
The straight line passes through the points (−0.7, 0) and (0.21, 0.45)

Using this information,

(b) find a complete equation for the model in the form

$$T = al^b$$

giving the value of a and the value of b, each to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant a.

(1)

$T = al^b$

a) logging both sides :

Question 10 continued

$$\log_{10} T = \log_{10} al^b$$

using $\log a + \log b = \log ab$

$$\log_{10} T = \log_{10} a + \log_{10} l^b$$

using $\log a^b = b \log a$

$$\log_{10} T = \log_{10} a + b \log_{10} l$$

b) $\log_{10} T = \log_{10} a + b\log_{10} l$ has a linear relationship, where b is the gradient

$$\therefore b = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{0.45 - 0}{0.21 - (-0.7)}$$

$$= \frac{0.45}{0.28}$$

$$= 1.607$$

$= 1.61$ to 3 s.f.

at $(-0.7, 0)$:

$$0 = \log_{10} a + 1.607 \times (-0.7)$$

$$\log_{10} a = 1.607 \times 0.7$$

$$\log_{10} a = \frac{9}{8}$$

Question 10 continued

$a = 10^{9/8}$

$= 13.335$

≈ 13.3 to 3 s.f.

c) acceleration at the time of release

11.

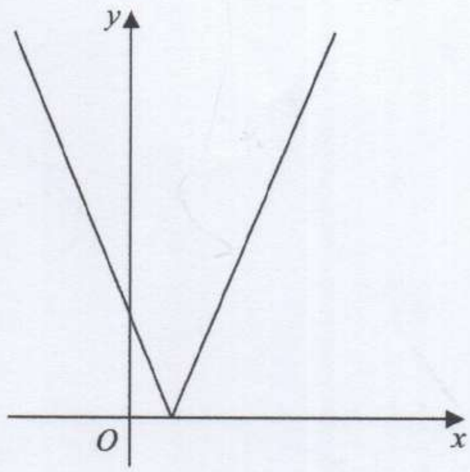


Figure 4

Figure 4 shows a sketch of the graph with equation

$$y = |2x - 3k|$$

where k is a positive constant.

(a) Sketch the graph with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = k - |2x - 3k|$$

stating

- the coordinates of the maximum point
- the coordinates of any points where the graph cuts the coordinate axes

(4)

(b) Find, in terms of k, the set of values of x for which

$$k - |2x - 3k| > x - k$$

giving your answer in set notation.

(4)

(c) Find, in terms of k, the coordinates of the minimum point of the graph with equation

$$y = 3 - 5\mathrm{f}\left(\frac{1}{2}x\right)$$

(2)

x-intercepts : at $f(x) = 0$

$k - |2x - 3k| = 0$

Question 11 continued

$$|2x - 3k| = k$$

either $2x - 3k = k$ or $-(2x-3k) = k$

$2x = k + 3k$ | $-2x + 3k = k$

$2x = 4k$ | $2x = 2k$

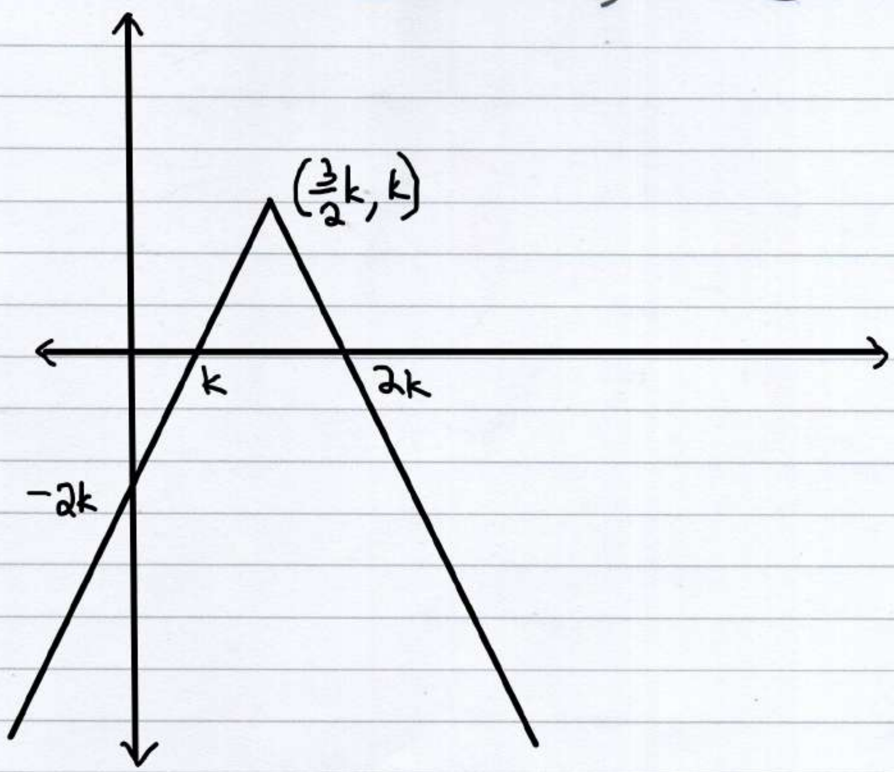
$x = 2k$ | $x = k$

y-intercept, at $x = 0$:

$$k - |-3k| = f(x)$$

$$f(x) = k - 3k$$
$$= -2k$$

maximum pt: $\left(\frac{2k+k}{2}, k\right) = \left(\frac{3}{2}k, k\right)$



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Question 11 continued

b) $k - |2x - 3k| > x - k$

$|2x - 3k| < 2k - x$

either $2x - 3k < 2k - x$ or $-(2x-3k) < 2k - x$

$3x < 5k$ | $-2x + 3k < 2k - x$

$x < \frac{5}{3}k$ | $x > k$

$\therefore \{x : k < x < \frac{5}{3}k\}$

c) $f(\frac{1}{2}x) = k - |x - 3k|$

max. value of $f(\frac{1}{2}x) = k$

$\therefore$ min. value of $y = 3 - 5k$

$\because$ stretch factor // to x-axis by 2

co-ordinates are $(2 \times \frac{3}{2}k, 3 - 5k)$

$= (3k, 3 - 5k)$

(Total for Question 11 is 10 marks)

12. (a) Use the substitution $u = 1 + \sqrt{x}$ to show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = \int_p^q \frac{2(u-1)^3}{u}\,du$$

where p and q are constants to be found.

(3)

(b) Hence show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,dx = A - B\ln 5$$

where A and B are constants to be found.

(4)

(a) given, $u = 1 + x^{1/2} \Rightarrow x^{1/2} = u - 1 \Rightarrow x = (u-1)^2$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$ upper limit $= 1 + 16^{1/2}$

$= 1 + 4$

$= 5$

$\frac{du}{dx} = \frac{1}{2x^{1/2}}$ lower limit $= 1 + 0^{1/2}$

$= 1$

$dx = 2x^{1/2}\,.du$

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\,.dx = \int_1^5 \frac{x}{u}\,.\,2x^{1/2}\,.du$$

$$= \int_1^5 \frac{(u-1)^2}{u}\,.\,2(u-1)\,.du$$

$$= \int_1^5 \frac{2(u-1)^3}{u}\,.du$$

$\therefore$ $q = 5$

$p = 1$

P 6 8 7 3 2 A 0 3 4 4 8

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Question 12 continued

b) $\int_1^5 \frac{2}{u}(u-1)^3 \cdot du$ $\qquad (u-1)^3 = u^3 - 3u^2 + 3u - 1$

$= \int_1^5 \frac{2}{u}(u^3 - 3u^2 + 3u - 1) \cdot du$

$= \int_1^5 \frac{2u^3 - 6u^2 + 6u - 2}{u} \cdot du$

$= \int_1^5 2u^2 - 6u + 6 - \frac{2}{u} \cdot du$

$= \left[\frac{2}{3}u^3 - \frac{6}{2}u^2 + 6u - 2\ln u\right]_1^5$

$= \left[\frac{2}{3}(5)^3 - 3(5)^2 + 6\times 5 - 2\ln 5\right] - \left[\frac{2}{3}(1)^3 - 3(1)^2 + 6(1) - 2\ln 1\right]$

$= \left[\frac{250}{3} - 75 + 30 - 2\ln 5\right] - \left[\frac{2}{3} - 3 + 6 - 0\right]$

$= \frac{250}{3} - 75 + 30 - 2\ln 5 - \frac{2}{3} - 3$

$= \frac{248}{3} - 48 - 2\ln 5$

$= \frac{104}{3} - 2\ln 5$

$A = \frac{104}{3}$ $\qquad B = -2$

(Total for Question 12 is 7 marks)

13. The curve C has parametric equations

$$x = \sin 2\theta \qquad y = \operatorname{cosec}^3\theta \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of θ **(3)**

(b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$ **(3)**

a) $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

$\frac{dx}{d\theta} = 2\cos 2\theta$

$y = \frac{1}{\sin^3\theta}$

Let $u = \sin\theta$ $\qquad \frac{du}{d\theta} = \cos\theta$

$\therefore \; y = \frac{1}{u^3} = u^{-3}$ $\qquad \frac{dy}{du} = -3u^{-4}$

$= -\frac{3}{u^4}$

$\frac{dy}{d\theta} = \frac{du}{d\theta} \times \frac{dy}{du}$

$= \cos\theta \times \frac{-3}{\sin^4\theta}$

$= \frac{-3\cos\theta}{\sin^4\theta}$

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14.

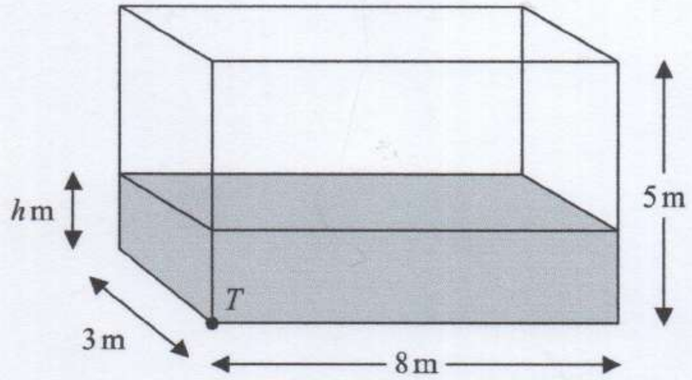

Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres
- water is flowing into the tank at a constant rate of $0.48\,\text{m}^3$ per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h\,\text{m}^3$ per minute

(a) Show that, according to the model,

$$1200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + B\mathrm{e}^{-kt}$$

where A, B and k are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

a) Given, $\frac{dV}{dt} = 0{\cdot}48 - 0{\cdot}1h$

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DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 14 continued

$$V \text{ at time } t = 3 \times 8 \times h$$
$$= 24h$$

$$\frac{dV}{dh} = 24$$

using chain rule,

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$0.48 - 0.1h = 24 \times \frac{dh}{dt}$$

$$4.8 - h = 240 \times \frac{dh}{dt}$$

$$24 - 5h = 1200 \frac{dh}{dt}$$

$$1200 \frac{dh}{dt} = 24 - 5h \qquad \text{Q.E.D.}$$

b) $1200 \frac{dh}{dt} = 24 - 5h$

$$\int \frac{1200}{24-5h} \, . \, dh = \int 1 . \, dt$$

$$\frac{1200 \ln|24-5h|}{-5} = t + c$$

Question 14 continued

Given,

at $t=0$, $h=2$

$$\frac{1200 \ln|24-5\times 2|}{-5} = c + 0$$

$$-240 \ln|24-10| = c$$

$$c = -240 \ln 14$$

$$\therefore \quad -240 \ln|24-5h| = t - 240 \ln 14$$

$$\ln|24-5h| = \frac{-t}{240} + \ln 14$$

$$24-5h = e^{\frac{-t}{240} + \ln 14}$$

$$24-5h = e^{\frac{-t}{240}} \times e^{\ln 14}$$

$$24-5h = 14e^{-t/240}$$

$$5h = 24 - 14e^{-t/240}$$

$$h = 4.8 - \frac{14}{5}e^{-t/240}$$

$\therefore \quad A = 4.8, \quad B = -\frac{14}{5}, \quad k = \frac{-1}{240}$

c) As $t \to \infty$, $-\frac{14}{5}e^{-t/240}$ tends to 0

this means that for large values of t, the height tends towards 4.8 metres

Thus the tank never becomes full.

(Total for Question 14 is 12 marks)

15. (a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and the value of α in radians to 3 decimal places.

(3)

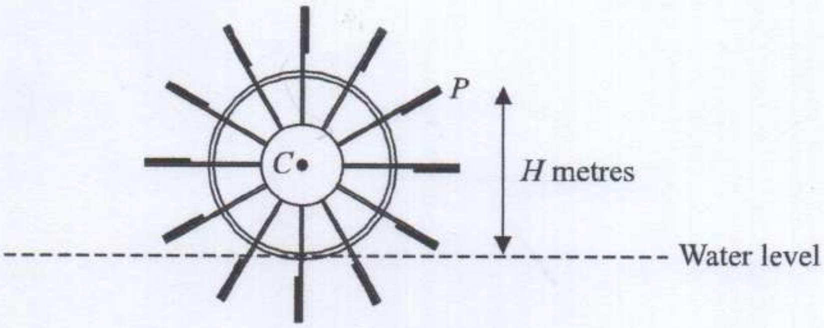


Figure 6

Figure 6 shows the cross-section of a water wheel.

The wheel is free to rotate about a fixed axis through the point C.

The point P is at the end of one of the paddles of the wheel, as shown in Figure 6.

The water level is assumed to be horizontal and of constant height.

The vertical height, H metres, of P above the water level is modelled by the equation

$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$$

where t is the time in seconds after the wheel starts rotating.

Using the model, find

(b) (i) the maximum height of P above the water level,

(ii) the value of t when this maximum height first occurs, giving your answer to one decimal place.

(3)

In a single revolution of the wheel, P is below the water level for a total of T seconds.

According to the model,

(c) find the value of T giving your answer to 3 significant figures.

(*Solutions based entirely on calculator technology are not acceptable.*)

(4)

In reality, the water level may not be of constant height.

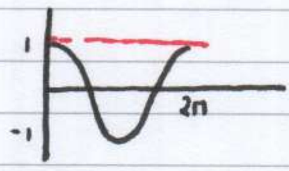
(d) Explain how the equation of the model should be refined to take this into account.

(1)

Question 15 continued

$\therefore \quad \cos(\theta + 0.464) = 1$

$\theta + 0.464 = 0, 2\pi$



$\theta = -0.464, 5.8195$

but $\theta = 0.5t$

$0.5t = -0.464, 5.8195$

$t = \cancel{-0.928}\ -0.928, 11.639$

but time cannot be negative

$\therefore$ $t = 11.639$ seconds

c) for T seconds, $H < 0$

$3 + 4\cos\theta - 2\sin\theta < 0$

$3 + 2\sqrt{3}\cos(\theta + 0.464) < 0$

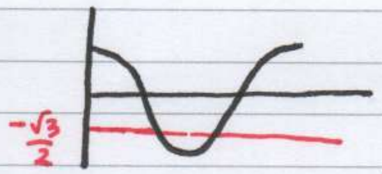
$2\sqrt{3}\cos(\theta + 0.464) < -3$

Question 15 continued

$$\cos(\theta + 0.464) < \frac{-3}{2\sqrt{3}}$$

$$\cos(\theta + 0.464) < -\frac{\sqrt{3}}{2}$$

Looking at cos graph,



values at which $\cos(\theta + 0.464) = -\frac{\sqrt{3}}{2}$: $\frac{5}{6}\pi$, $\frac{7}{6}\pi$

$$\therefore \quad \frac{5\pi}{6} < \theta + 0.464 < \frac{7}{6}\pi$$

$$2.154 < \theta < 3.2012$$

but $\theta = 0.5t$,

$$2.154 < 0.5t < 3.201$$
$$4.308 < t < 6.402$$

$\approx$ $4.31 < t < 6.40$ to 3 s.f.

d) model the water level using a trigonometric equation